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K-Theoretic Invariants for C^*-Algebras Associated to Transformations and Induced Flows

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We discuss a technique of studying the K-theory of a unital C^*-algebra associated to a homomorphism on a compact metric space $(Y, \mathbb{Z})$ by examining the non-unital C^*-algebra associated to the induced topological flow $(\mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}}(Y), \mathbb{R})$. The Thom isomorphism of Connes and the Schwartzman asymptotic cycle are used to calculate the range of the trace corresponding to an invariant measure on the K_0 group of $C^*(X, \mathbb{R})$ for a continuous flow on a compact metric space $(X, \mathbb{R})$. Under certain conditions projections in $C^*(X, \mathbb{R})$ with trace r corresponding to cross sections to the flow can be constructed for every positive real number r in this range, again by combining techniques of Connes and Schwartzman. Applications to the calculation of the tracial range of $K_0(C^*(Y, \mathbb{Z}))$ are discussed. In particular, this invariant is calculated for minimal affine actions of $\mathbb{Z}$ on n-tori which have quasi-discrete spectrum, and for minimal actions of $\mathbb{Z}$ on compact abelian groups with topologically discrete spectrum. In both cases this tracial invariant is shown to be the preimage of the eigenvalues for $(Y, \mathbb{Z})$ under the natural projection $\phi: \mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$. © 1986 Academic Press, Inc.

INTRODUCTION

In the study of transformation group C^*-algebras, the range of a trace corresponding to an invariant measure on the K_0 group of the algebra has proved to be a key isomorphism invariant for certain algebras [7, 17, 19]. The purpose of this paper is to show how Connes' results on crossed products by $\mathbb{R}$ can be employed in the calculation of this range for crossed products by $\mathbb{Z}$. In particular we are able to calculate this invariant for C^*-algebras corresponding to ergodic affine actions of $\mathbb{Z}$ on the n-torus which have quasi-discrete spectrum. As such actions are generalizations of $\mathbb{Z}$-actions on the circle with pure point spectrum, they are natural candidates for study, since the corresponding C^*-algebras are generalizations of the

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irrational rotation algebra of [19]. For the irrational rotation algebras, the range of the trace was first computed (and shown to be a complete isomorphism invariant) in [31] and [17], where the Rieffel projections were shown to give a complete set of generators for the K_0 group. The method which we use to calculate the trace of the algebras under study does not use the Pimsner–Voiculescu exact sequence but instead relies heavily on the results of Connes [4]. Thus instead of examining crossed products by actions of $\mathbb{Z}$ we examine crossed products by induced actions of $\mathbb{R}$, a method which has been shown to be fruitful in the past [4, 11]. Connes' formula for the tracial range is relatively easy to use in our case, since the actions of $\mathbb{R}$ induced from our class of affine actions on n-tori turn out to be one-parameter subgroup flows on nilmanifolds $\Gamma\backslash N$, where N is a simply-connected nilpotent Lie group of dimension $n+1$ and Γ is a cocompact discrete subgroup [see 28].

A simple application of Corollary 2 of [4] allows us to prove

THEOREM 3.3. *Let $(T^n, \mathbb{Z})$ be a minimal action of $\mathbb{Z}$ on T^n which has topologically quasi-discrete spectrum. Then $\tau_\nu(K(C^*(T^n, \mathbb{Z}))) = \phi^{-1}(E)$ where ν is Haar measure, and where $E \subset S^1$ is the set of eigenvalues for the action of $\mathbb{Z}$ on T^n and $\phi\colon \mathbb{R} \to S^1$ is projection* mod $\mathbb{Z}$.

A similar application of Connes' Corollary 2 allows us to prove by other methods the following result of N. Riedel [18]:

THEOREM 2.5. *Let $(Y, \nu, \mathbb{Z})$ be a minimal action of $\mathbb{Z}$ by automorphisms on the compact metric space Y which has topologically discrete spectrum. Then*

$$\tau_\nu(K_0(C^*(Y, \mathbb{Z}))) = \phi^{-1}(E(Y, \mathbb{Z})),$$

where $E(Y, \mathbb{Z})$ is the set of eigenvalues for $(Y, \mathbb{Z})$.

The significance of the eigenvalues for an action of $\mathbb{Z}$ in the K_0 group of the corresponding C^*-algebra is evident form the theorems above. From the point of view of this paper the significance can be explained in an intuitive fashion which we briefly outline here. Each eigenvalue of $(Y, \mathbb{Z})$ induces a coutable number of eigencharacters of the induced action $(X, \mathbb{R})$. Each eigencharacter for $(X, \mathbb{R})$ which is positive rise to a cross-section transversal to the flow, K_β [24]. The corresponding cross-sections give rise to projections in $C^*(X, \mathbb{R})$, following the construction of Connes [5], who used "transversals." Via Rieffel's concept of strong Morita equivalence [cf. 19] these projections correspond to projections in $M_j(C^*(Y, \mathbb{Z}))$ for some natural number j and hence provide a map of $K_0(C^*(K_\beta))$ into $K_0(C^*(Y, \mathbb{Z}))$. If one takes the eigenvalue 1 for $(Y, \mathbb{Z})$, one of the corresponding cross-sections to $(X, \mathbb{R})$ can be identified with Y and we thus obtain the natural homomorphism of $K_0(C(Y))$ into $K_0(C^*(Y, \mathbb{Z}))$ discussed in [17].

The correspondence between the positive portion of $\phi^{-1}(E(Y, \mathbb{Z}))$ and $\tau_\nu(K_0(Y, \mathbb{Z}))$ is in general not surjective (e.g., if Y is not connected). However, for many of the examples with which we are concerned, surjectivity fortuitously holds, and gives a way of explaining the importance of eigenfunctions in constructing projections and establishing invariants for strong Morita equivalence.

The paper is divided into four sections; the first reviews flows which are induced from a transformation or built under a function, cross-sections, and the strong Morita equivalence of the C^*-algebras and their traces. In the second section we discuss a version of Connes' Corollary 2 of [4], expressed in terms of the asymptotic cycle A_μ of Schwartzman [24]. This version takes the form

$$\tau_\mu^*(K_0(C^*(X, \mathbb{R}))) = A_\mu(H^1(X, \mathbb{Z})),$$

where $(X, \mathbb{R})$ is a flow on a compact separable metric space, μ is an $\mathbb{R}$-invariant probability measure on X, and $H^1(X, \mathbb{Z})$ is Cech cohomology. The results of Schwartzman relating cross sections to the positive part of the range of A_μ can, in light of Connes' techniques of [5], be used to provide the formula for the trace of the corresponding projection. For $[f] \in H^1(X, \mathbb{Z})$ with f continuously differentiable, $(1/2\pi i) f'(x) \overline{f(x)} > 0$, $\forall x \in X$, this takes the form $\tau_\mu(P_{K_f}) = \int_X (1/2\pi i) f'(x) \overline{f(x)}\, d\mu$ where K_f is the cross section to the flow $(X, \mathbb{R})$ defined by $K_f = \{x \in X | f(x) = 1\}$, and where P_{K_f} is the Connes projection associated to K_f. The proof of Riedel's theorem is an immediate consequence. The third section establishes the range of the trace on the K_0 group of C^*-algebras corresponding to affine actions. In the final section we concentrate on a particular family of examples, Anzai's skew products on the torus, and use the methods established in the previous section together with ideas suggested in [19] to calculate the isomorphism and strong Morita equivalence invariants.

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1. Actions of $\mathbb{Z}$ and the Induced Actions of $\mathbb{R}$

We suppose that we are given an action $(Y, \mathbb{Z})$ where Y is the compact metric space and $\mathbb{Z}$ acts freely on Y as a group of topological automorphisms. Suppose that $f\colon Y \to \mathbb{R}^+$ is a continuous function taking on positive values. Then as described in [15], we can define a one cocycle $\alpha_f\colon Y \times \mathbb{Z} \to \mathbb{R}$ by setting $\alpha_f(y, 1) = f(y)$ and extending to all $Y \times \mathbb{Z}$ in such a way that the cocycle identify always holds. Then $\mathbb{R}$ and $\mathbb{Z}$ both act on the product space $Y \times \mathbb{R}$ by setting

$$(y, r) \cdot n = (y \cdot n, r - \alpha_f(y, n)),$$
$$t \cdot (y, r) = (y, t + r), \qquad n \in \mathbb{Z}, t \in \mathbb{R}$$

These two actions commute and hence there is an action of $\mathbb{R}$ on the quotient space $(Y, \mathbb{R})/\mathbb{Z}$. The action of $\mathbb{R}$ so obtained produces what is commonly called *the flow built under the function* f [1]. The space X is also a compact metric space and if ν_Y is a finite $\mathbb{Z}$-invariant measure on Y, there is a corresponding $\mathbb{R}$-invariant measure on X obtained by setting

$$\mu_X = \Pi_* \left(\int \chi_{G_f} \, d(\mu_Y \times m) \right),$$

where $G_f = \{(y, r) \in Y \times \mathbb{R} \mid 0 \leqslant r \leqslant f(y)\}$, m is Lebesgue measure on $\mathbb{R}$, and $\Pi\colon Y \times \mathbb{R} \to X$ is the quotient identification.

If instead of a positive f one chooses a function which is negative, then the cocycle α_f may be formed as before, along with action of $\mathbb{Z}$ and $\mathbb{R}$ on $Y \times \mathbb{R}$. Again the space X of $\mathbb{Z}$-orbits is a compact metric space on which a flow $(X, \mathbb{R})$ may be defined. An invariant measure can be determined in a very similar fashion to the μ for positive f. We examine negative functions because when $f \equiv -1$ on Y, the corresponding "flow built over the function" is a standard construction termed the *action of* $\mathbb{R}$ *induced from* $(Y, \mathbb{Z})$, which we shall denote by $(\mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}}(Y), \mathbb{R})$. The general construction of induction from an action of a closed subgroup to an action of an arbitrary locally compact group containing it as a closed subgroup is described in more detail in [15, 27]. If ν_Y is a $\mathbb{Z}$-invariant probability measure on Y then, as described above, a $\mathbb{R}$ invariant measure μ_X on X can be constructed as follows. Let $\theta\colon \mathbb{R}/\mathbb{Z} = S^1 \to \mathbb{R}$ be the Borel section mapping $e^{2\pi i x}$ to x, $\forall x \in [0, 1)$, and let $\beta\colon \mathbb{R}/\mathbb{Z} \times \mathbb{R} \to \mathbb{Z}$ be the cocycle defined $m_{S^1} \times m_{\mathbb{R}}$ almost everywhere by

$$\beta(z, r) = \theta(z \cdot 2^{2\pi i r}) - r + \theta(z).$$

Then $\mathbb{R}$ acts as a group of Borel transformations on $(S^1 \times_\beta Y, m_{S^1} \times \nu_Y)$ via the skew product construction,

$$(z, y) \cdot r = (z\, e^{2\pi i r}, y \cdot \beta(y, r))$$

Furthermore the map $\phi: (S^1 \times_\beta Y, m_{S^1} \times \nu_Y, \mathbb{R}) \to (\mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}}(Y), \phi_*(m_{S^1} \times \nu_Y), \mathbb{R})$ defined by $\phi(z, y) = \pi(y, \theta(z))$, where $\pi: Y \times \mathbb{R} \to \mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}}(Y)$ is the quotient identification, is a measure-preserving $\mathbb{R}$-equivalent Borel isomorphism (but not necessarily a topological conjugacy). Hence $\phi_*(m_{S_1} \times \nu_Y)$ is a finite $\mathbb{R}$-invariant measure on X whose total measure is equal to $\nu_Y(Y) = 1$, since ν_Y is a probability measure. It is clear that this measure can be identified with $\mu_X = \pi^* \int \chi_{G_f}(m \times \nu_Y)$ described earlier.

The method of induction and that of building the flow under a function allow one to obtain flows (actions of the real line on topological spaces) from topological transformations (actions of the integers). Going in the reverse direction, the notion of transversal to a flow gives a method of constructing transformations from flows. The following definition is that of Schwartzman [24], whose terminology we follow:

DEFINITION 1.1. Let $(X, \mathbb{R})$ be a flow where X is a compact metric space. A subset K of X is said to be a *cross section* to the flow if K is closed and the mapping $\psi: K \times \mathbb{R} \to X$ sending (k, t) to kt is a local homeomorphism onto all of X.

The local homeomorphism condition and the compactness of K imply that the map $f_K: K \to \mathbb{R}^+$ defined by

$$f_K(k) = \inf\{r > 0 \mid kr \in K\}$$

is a continuous function on K bounded from above and bounded away from 0. Thus $T: K \to K$ defined by $T(k) = kf_K(k)$ is a homeomorphism of K onto itself which provides an action $(K, \mathbb{Z})$ of $\mathbb{Z}$ on K. Let $(X, \mathbb{R})$ be the flow built under the function f_K on $(K, \mathbb{Z})$, where $\Pi: K \times \mathbb{R} \to X$ is the quotient map. Then since ψ and Π are quotient maps with the same fibers, the map $\psi \circ \Pi^{-1}: X_1 \to X$ is well defined and provides an $\mathbb{R}$-equivariant homeomorphism between $(X_1, \mathbb{R})$ and $(X, \mathbb{R})$. Thus for any cross section K of $(X, \mathbb{R})$ there exists an action $(K, \mathbb{Z})$ of $\mathbb{Z}$ on K by transformations and a continuous positive function f on K so that $(X, \mathbb{R})$ is the flow built under the function f on $(K, \mathbb{Z})$.

The above constructions have been shown to be very fruitful in the study of the corresponding operator algebras. The main reason for this is the strong Morita equivalence of the C^*-algebra of a transformation and the C^*-algebra of a corresponding flow built under a function or an induced flow, as mentioned in [4, 20, 21]. This allows one to obtain knowledge of properties about the $(Y, \mathbb{Z})$ algebra from knowledge about the $(X, \mathbb{R})$ algebra, or vice versa. In studying the K-theory of the $(Y, \mathbb{Z})$ algebra, one can apply the Pimsner–Voiculescu exact sequence or Connes' Thom isomorphism depending on which track one desires to take. This method of inducing has previously been employed in the study of transformation group C^* algebras in [4, 11, 21].

Let $(Y, \mathbb{Z})$ be an action where Y is a compact metric space and let $f\colon Y \to \mathbb{R}$ be a continuous strictly positive or strictly negative function. Form the flow under or over the function f which is positive or negative, respectively, and denote it by $(X, \mathbb{R})$. We briefly review the construction of the C^*-algebras $C^*(X, \mathbb{R})$ and $C^*(Y, \mathbb{Z})$ corresponding to $(X, \mathbb{R})$ and $(Y, \mathbb{Z})$.

If (M, G) is the action (on the right) of a locally compact group G on M we denote by $C_c(M, G)$ the set of continuous functions on $M \times G$ with compact support. Then $C_c(M, G)$ has the structure of a normed $*$-algebra as follows:

$$f^*(m, g) = \overline{f(mg, g^{-1})},$$

$$f \cdot h(m, g) = \int_G f(m, g_1)\, h(mg_1, g_1^{-1} g)\, dg_1,$$

$$\|f\| = \sup(\|\pi(f)\|),$$

where the supremum is taken over all continuous representations π of the Banach $*$-algebra $L^1(G, C^*(M))$ containing $C_c(M, G)$ as a dense $*$-subalgebra. Let $C^*(M, G)$ denote the C^*-algebraic completion of $C_c(M, G)$; it is termed the *C^*-algebra corresponding to* (M, G).

If we set $A = C_c(Y, \mathbb{Z})$ and $B = C_c(X, \mathbb{R})$ then by definition A and B are norm-dense $*$-subalgebras of $C^*(Y, \mathbb{Z})$ and $C^*(X, \mathbb{R})$, and they are particularly useful in that the set $C_c(Y \times \mathbb{R}) = C$ is defined as an $A-B$ equivalence bimodule by setting

$$(f \circ \xi)(y, r) = \sum_{j \in \mathbb{Z}} f(y, j)\, \xi(y \cdot j, r - \alpha_f(y, j)),$$
$$f \in A,\ \xi \in C,\ (y, r) \in Y \times \mathbb{R}, \tag{1}$$

$$(\eta \circ g)(y, r) = \int_{\mathbb{R}} \eta(y, l)\, g(\pi(y, r), r - l)\, dl, \qquad \eta \in C,\ g \in B, \tag{2}$$

$$\langle \xi_1, \xi_2 \rangle_A (y, j) = \int_{\mathbb{R}} \xi_1(y, l)\, \overline{\xi_2(j, l - \alpha_f(y, l))}\, dl, \qquad \xi_1, \xi_2 \in C, \tag{3}$$

$$\langle \xi_1, \xi_2 \rangle_B (\Pi(y, r), t) = \sum_{j \in \mathbb{Z}} \overline{\xi_1(y \cdot j, r + \alpha_f(y, j))}\, \varepsilon_2(yj, r + t + \alpha_f(y, j)),$$
$$\xi_1, \xi_2 \in C. \tag{4}$$

Note that the function of $Y \times \mathbb{R}$ on the right-hand side is $\mathbb{Z}$-invariant and hence is the lift of some function on X. We note also that our formulas are different from those of [21] since our group actions are written on the right, not on the left as in [21].

The results of Green [11] and Rieffel [21] show that C, suitably structured and completed, provides a strong Morita equivalence bimodule between $\bar{A}=C^*(Y,\mathbb{Z})$ and $\bar{B}=C^*(X,\mathbb{R})$; in other words, as defined in [20] there exists a left-$\bar{A}$ right-$\bar{B}$ bimodule $\bar{C}$ having $\bar{A}$ and $\bar{B}$ valued inner products satisfying

(1) $\langle x, y\rangle_{\bar{A}}\, z = x\langle y, z\rangle_{\bar{B}} \qquad x, y, z\in \bar{C}$.

(2) The representation of $\bar{A}$ (resp. $\bar{B}$) on $\bar{C}$ is a continuous representation by operators which are bounded for $\langle\ ,\ \rangle_{\bar{B}}$ (resp. $\langle\ ,\ \rangle_{\bar{A}}$).

(3) The linear span of $\langle \bar{C}, \bar{C}\rangle_{\bar{A}}$ is dense in $\bar{A}$ and similarly for $\langle \bar{C}, \bar{C}\rangle_{\bar{B}}$ in $\bar{B}$.

By using a construction due to Connes [5] we can see that the Morita equivalence is in fact implemented by a projection in A:

LEMMA 1.2. *Let $(Y,\mathbb{Z})$ be an action of $\mathbb{Z}$ on the compact metric space Y, and let $(X,\mathbb{R})$ be the flow built under or over the strictly positive or strictly negative continuous function f. Then there exists a projection P_Y in $C^*(X,\mathbb{R})$ such that $P_Y C^*(X,\mathbb{R})\, P_Y$ is *-isomorphism to $C^*(Y,\mathbb{Z})$.*

Proof. The construction that follows is modelled on Connes' construction in [5], the only difference being that between the terminology of strong Morita equivalence and that of Hilbert C^*-modules.

Take $A=C_c(Y,\mathbb{Z})$, $B=C_c(X,\mathbb{R})$, and $C=C_c(Y\times\mathbb{R})$. Then as above C is a left-A and right-B module with A- and B-valued inner products. We wish to find $m\in C_c(Y\times\mathbb{R})$ such that $\langle m, m\rangle_A=\mathrm{Id}_{C^*(Y,\mathbb{Z})}$. It then follows, using the methods laid out in Proposition 2.1 of [19], that $\langle m, m\rangle_B=P_Y$ is a projection in B and furthermore that $C^*(Y,\mathbb{Z})$ can be embedded into $C^*(X,\mathbb{R})$ as the full corner $P_Y C^*(X,\mathbb{R})\, P_Y$. We note now that since f is bounded away from zero and does not change sign on the compact set Y, there exists $\varepsilon>0$ such that the map Π: $Y\times\mathbb{R}\to X$ restricted to $Y\times[-\varepsilon,\varepsilon]$ is one-to-one into X. Thus if $m(y,l)$ is a continuous function on $(Y\times\mathbb{R})$ with support contained in $Y\times[-\varepsilon,\varepsilon]$,

$$\langle m, m\rangle_A(y,j)=\int m(y,l)\,\overline{m(yj, l-\alpha(y,j))}\,dl$$

is identically zero for fixed $j\neq 0$, for in that case either the left term or the rigth term of the integrand is equal to zero. In particular if we set $m_Y(y,l)=\gamma(l)$, where $\gamma(l)$ is a smooth function on $\mathbb{R}$ with support contained in $[-\varepsilon,\varepsilon]$ such that $\int_{\mathbb{R}}|\gamma(l)|^2\,dl=1$, we obtain

$$\langle m_Y, m_Y\rangle_A(y,j)=\begin{cases}0, & j\neq 0,\ \forall y\in Y,\\ \int|\gamma(l)|^2\,dl=1, & j=0,\ \forall y\in Y.\end{cases}$$

Thus $\langle m_Y, m_Y\rangle_A$ is equal to Id_A, so that $\langle m_Y, m_Y\rangle_B = P_Y$ is the desired projection in $C^*(X, \mathbb{R})$. ∎

Remark 1.3. The above lemma provides for any cross section Y to the flow $(X, \mathbb{R})$ an injection of $C^*(Y)$ into $C^*(X, \mathbb{R})$, and this injection induces the map $\phi!\colon K_0(C^*(Y)) \to K_0(C^*(X, \mathbb{R}))$ described by Connes in Chap. 8 of [5] for transversals. In particular, $\phi!([\mathrm{Id}_{C(Y)}]) = [\langle m_Y, m_Y\rangle_B]$.

Fix now an action $(Y, \mathbb{Z})$ where Y is a compact metric space, with ν_Y as a $\mathbb{Z}$-invariant Borel probability measure. Let $f\colon Y \to \mathbb{R}$ be a strictly positive or strictly negative continuous function. As mentioned above the flow $(X, \mathbb{R})$ constructed from f carries a finite Borel $\mathbb{R}$-invariant measure which corresponds, roughly speaking, to the product measure $\nu_Y \times m$ applied to the "space beneath (or above) that graph of f," where f is strictly positive or negative, respectively. If we denote this measure by μ_X, then associated to ν_Y an μ_X are the finite traces τ_{ν_Y} and τ_{μ_X} on $C^*(Y, \mathbb{Z})$ and $C^*(X, \mathbb{R})$, respectively, which can be calculated on the norm dense *-subalgebras A and B by

$$\tau_{\nu_Y}(g(y, j)) = \int_Y g(y, 0)\, d\nu_Y,$$

$$\tau_{\mu_X}(h(x, t)) = \int_X h(x, 0)\, d\mu_X.$$

As has been noted previously by others [10, 19], the strong Morita equivalence bimodule established between $C^*(X, \mathbb{R})$ and $C^*(Y, \mathbb{Z})$ allows one to induce the trace τ_{ν_Y} to a trace $\mathrm{Ind}(\tau_{\nu_Y})$ on $C^*(X, \mathbb{R})$. Indeed if we denote by $\bar{C}$ the imprimitivity bimodule between $\bar{A} = C^*(Y, \mathbb{Z})$ and $\bar{B} = C^*(Y, \mathbb{Z})$, then

$$\mathrm{Ind}(\tau_{\nu_Y})(\langle f, g\rangle_{\bar{B}}) = \tau_{\nu_Y}(\langle g, f\rangle_{\bar{A}}).$$

The following proposition is certainly known, implicity stated in [10] and is used in [5], but as we have not found it explicitly stated in the literature we include its proof here.

PROPOSITION 1.4. *Let $(Y, \mathbb{Z})$ be an action of $\mathbb{Z}$ on the compact separable metric space Y, and let $(X_f, \mathbb{R})$ be the flow associated to a strictly positive or negative continuous function f on Y. Let ν_Y be a $\mathbb{Z}$-invariant Borel probability measure on Y and μ_X the corresponding finite $\mathbb{R}$-invariant Borel measure on X with τ_{ν_Y} and τ_{μ_X} the traces on the associated C^*-algebras. If $\bar{C}$ is the $C^*(Y, \mathbb{Z})$–$C^*(X, \mathbb{R})$ equivalence bimodule described in Lemma* 1.2, *then*

$$\mathrm{Ind}_{\bar{C}}(\tau_{\nu_Y}) = \tau_{\mu_X}.$$

Proof. We consider $f > 0$; the case $f < 0$ is similar. Recall that $\bar{C}$ is the completion of $C_c(Y \times \mathbb{R})$ carrying the $C_c(Y, \mathbb{Z})$- and $C_c(X, \mathbb{R})$-valued inner products as previously defined. We verify that $\operatorname{Ind}_{\bar{C}}(\tau_{\nu_Y}) = \tau_{\mu_X}$ on $\langle C_c(Y \times \mathbb{R}), C_c(Y \times \mathbb{R}) \rangle_{C_c(X, \mathbb{R})}$. By using continuity of the trace one obtains the equality for all of $C^*(X, \mathbb{R})$.

$$\begin{aligned}
&\operatorname{Ind}_{\bar{C}}(\tau_{\nu_Y})(\langle h, g \rangle_{C_c(X, \mathbb{R})}) \\
&\quad = \tau_{\nu_Y}(\langle g, h \rangle_{C_c(Y, \mathbb{Z})}) \\
&\quad = \int_Y \int_{\mathbb{R}} g(y, l)\, \overline{h(y, l)}\, dl\, d\nu_Y
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\tau_{\mu_X}(\langle h, g \rangle_{C_c(X, \mathbb{R})}) \\
&\quad = \tau_{\mu_X}(\langle h, g \rangle(\Pi(y, r), t)) \\
&\quad = \int_Y \int_0^{f(y)} \langle h, g \rangle(\Pi(y, r), 0)\, dr\, d\nu_Y \\
&\quad = \int_Y \int_0^{f(y)} \left(\sum_j h(yj, r + \alpha_f(y, j))\, g(yj, r + \alpha_f(y, j)) \right) dr\, d\nu_Y \\
&\quad = \int_Y \int_0^{f(y)} \left(\sum_j \overline{h(yj, r - \alpha_f(yj, -j))}\, g(yj, r - \alpha_f(yj, -j)) \right) dr\, d\nu_Y \\
&\quad = \int_Y \sum_j \int_{\alpha_f(yj, -j)}^{\alpha_f(yj, -j) + f(y)} \overline{h(yj, r)}\, g(yj, r)\, dr\, d\nu_Y \\
&\quad = \int_Y \sum_j \int_{\alpha_f(yj, -j)}^{\alpha_f(yj, -j) + \alpha_f(y, 1)} \overline{h(yj, r)}\, g(yj, r)\, dr\, d\nu_Y \\
&\quad = (\text{using } \mathbb{Z}\text{-invariance of } \nu_Y) \\
&\qquad \int_Y \sum_j \int_{\alpha_f(y, -j)}^{\alpha_f(y, -j+1)} \overline{h(y, r)}\, g(y, r)\, dr\, d\nu_Y.
\end{aligned}$$

If M_f and m_f are the maximum and minimum of $|f|$ on Y, respectively, we note that

$$|j|\, M_f \geqslant |\alpha_f(y, j)| \geqslant |j|\, m_f, \qquad \forall y \in Y.$$

We thus obtain

$$\begin{aligned}
&\tau_{\mu_X}(\langle h, g \rangle_{C_c(X, \mathbb{R})}) \\
&\quad = \int_Y \int_{\mathbb{R}} \overline{h(y, r)}\, g(y, r)\, dr\, d\nu_Y \\
&\quad = \operatorname{Ind}(\tau_{\nu_Y})(\langle h, g \rangle_{C_c(X, \mathbb{R})}) \text{ as desired.} \quad \blacksquare
\end{aligned}$$

2. Eigenvalues and Projections

As mentioned in the Introduction, Schwartzman [24] made one of the first applications of algebraic topology to topological dynamics by associating to each flow $(X, \mathbb{R})$ with $\mathbb{R}$-invariant probability measure μ the "average asymptotic cycle," A_μ. This was a map of $H^1(X, \mathbb{Z})$ into $\mathbb{R}$, where $H^1(X, \mathbb{Z})$ denotes the elements of the first Cech cohomology group (with integer coefficients). The results of Connes imply that the range of A_μ is precisely $\tau_\mu(K_0(C^*(X, \mathbb{R}))$. Hence the map A_μ is especially useful in the study of $C^*(X, \mathbb{R})$, and can roughly speaking be regarded as the average value of maps $A_p: H^1(X, \mathbb{Z}) \to \mathbb{R}$, where $\{A_p \mid p \in G \subset X\}$ is a collection of maps which Schwartzman constructs for particular generic subset G of X. The notion of asymptotic cycles on manifolds was generalized by Ruelle and Sullivan [23], and their results were the ones employed in [4] and [5]. For our purposes the setting of [24] is enough, and we restrict ourselves to that case. We briefly summarize relevant results of [24].

Recall that $H^1(X, \mathbb{Z})$ can be regarded as $G(X)/E(X)$, where $G(X)$ represents the multiplicative group of continuous functions mapping X to $S^1 = \{z \mid |z| = 1\}$ and where $E(X)$ represents the subgroup of $G(X)$ consisting of those functions which can be expressed as $\exp(2\pi i\, H(X))$ for some continuous function $H: X \to \mathbb{R}$.

Let K be a fixed cross-section to the flow $(X, \mathbb{R})$, and define $f_K \in G(X)$ by setting

$$f_K(x) = \exp(2\pi i\, t_x/T_x), \qquad x \in X \tag{1}$$

where t_x is the least nonnegative real number with $x(-t_x) \in K$, and T_x is the least positive real number such that $x(T_x - t_x) \in K$. Conversely given $f \in G(X)$, under certain circumstances one can construct a cross-section K_f such that

$$[f] = [f_{K_f}] \qquad \text{in } H^1(X, \mathbb{Z}):$$

Theorem 2.1 [24]. *Let $f \in G(X)$. A necessary and sufficient condition that there exists a cross section K_f for which $[f(x)] = [f_{K_f}(x)] \bmod E(X)$ is that $A_\mu(f) > 0$ for every positive $\mathbb{R}$-invariant Borel measure μ on X.*

If f is a function which is continuously differentiable with respect to the flow and $(1/2\pi i) f'(x)\overline{f(x)} > 0$ $\forall x \in X$, then f satisfies the conditions of the theorem, and $K_f = \{x \in X \mid f(x) = 1\}$. This is true since then

$$0 < \frac{1}{2\pi i}\int_X f'(x)\overline{f(x)}\, d\mu = A_\mu([f]) \tag{2}$$

for every finite invariant measure μ, as is shown in [24, Sect. 4]. In fact if g satisfies the condition of Theorem 2.1, then there exists a function g_D which is differentiable with respect to the flow, with $(1/2\pi i)\, g'_D \bar{g}_D > 0$ $\forall x \in X$, and such that $[g] = [g_D] \bmod E(X)$ [24]. Hence if certain conditions are satisfied, one can associate to every element of $\mathbb{R}^+ \cap A_\mu(H^1(X, \mathbb{Z}))$ a cross section in $(X, \mathbb{R})$. In particular this is so if $(X, \mathbb{R})$ is uniquely ergodic so that there is only one $\mathbb{R}$-invariant probability measure on X. We note also that if $f_\alpha \colon X \to S^1$ is an eigenfunction with eigencharacter $\alpha > 0$, so that

$$f_\alpha(xt) = e^{2\pi i \alpha t} f_\alpha(x), \qquad \forall x \in X, \forall t \in \mathbb{R},$$

then Schwartzman's theorem associates the cross section

$$K_\alpha = \{x \mid f_\alpha(x) = 1\}.$$

We shall show that if P_{K_α} is the projection in $C^*(X, \mathbb{R})$ associated to K_α in the previous section, then

$$\tau_\mu(P_{K_\alpha}) = A_\mu(f_\alpha) = \alpha. \tag{3}$$

We recall that a particularly beautiful consequence of Connes' Thom Isomorphism theorem was the simple formula given for the calculation of $\tau_\mu(K_0(C^*(V, \mathbb{R})))$ for smooth flows $(V, \mathbb{R})$ on a compact manifold V with invariant probability measure μ: then

$$\tau_\mu(K_0(C^*(V, \mathbb{R}))) = \langle C, H^1(V, \mathbb{Z})\rangle, \tag{4}$$

where C is the Ruelle–Sullivan current defined by $C([\omega]) = \int_V \omega(X)\, d\mu$, for ω a smooth one form, where X is the smooth vector field associated to the flow. As the Ruelle–Sullivan current is a generalization of the Schwartzman cycle A_μ, Corollary 2 of [4] can actually be extended to include flows $(X, \mathbb{R})$ on a compact separable metric space X, and we restate it as follows:

THEOREM 2.2 [4]. *Let $(X, \mathbb{R})$ be a flow on the compact separable metric space X, and suppose that μ is a finite $\mathbb{R}$-invariant Borel measure on X. Then*

$$\tau_\mu(K_0(C^*(X, \mathbb{R}))) = \langle A_\mu, H^1(X, \mathbb{Z})\rangle,$$

where A_μ is the Schwartzman asymptotic cycle.

Proof. We note that Connes' proof (of Corollary 2 of [4]) comes from his Theorem 2 and from the fact that if a map $u \colon V \to U(n, \mathbb{C})$ is differentiable with respect to the flow, then

$$\operatorname{Tr}\left(\frac{1}{2\pi i} u'(v)\, u^*(v)\right) = \frac{1}{2\pi i} \det(u(v))' \, \overline{\det u(v)}. \tag{5}$$

But (5) is true also for $V = X =$ a compact metric space, as is Theorem 2 of [4]. Hence, denoting by ϕ_1 the Thom isomorphism $\phi_1: K_1(C^*(X)) \to K_0(C^*(X, \mathbb{R}))$, it follows that for $u: X \to U(n, \mathbb{C})$ differentiable with respect to the flow

$$\begin{aligned} \tau_\mu(\phi_1([u])) &= \frac{1}{2\pi i} \int \operatorname{Tr}(u'(x)\, u^*(x))\, d\mu \\ &= \frac{1}{2\pi i} \int (\det u(x))' \, \overline{\det u(x)}\, d\mu. \end{aligned} \tag{6}$$

Hence $\tau_\mu(K_0(C^*(X, \mathbb{R}))) \subseteq \langle A_\mu, H^1(X, \mathbb{Z}) \rangle$. In the other direction, if $\gamma \in H^1(X, \mathbb{Z})$ with continuously differentiable representative $f \in G(X)$, then

$$A_\mu(\gamma) = A_\mu([f]) = (\text{by } 2)$$
$$\frac{1}{2\pi i} \int_X f'(x) \overline{f(x)}\, d\mu = \tau_\mu(\phi_1(i([f]))),$$

where $i: H^1(X, \mathbb{Z}) \to K_1(C^*(X))$ is the natural inclusion. ∎

We see that if $\gamma \in H^1(X, \mathbb{Z})$ is represented by continuously differentiable f, then $\phi_1(i[f])$ is an abstract element of $K_0(C^*(X, \mathbb{R}))$ whose "trace" is equal to $(1/(2\pi i)) \int_X f'(x) \overline{f(x)}\, d\mu$. However, if f is such that $(1/2\pi i) f'(x) \overline{f(x)} > 0$, $\forall x \in X$, Theorem 2.1 of Schwartzman and the constructions given in the previous section allow us to explicitly construct a projection P in $C^*(X, \mathbb{R})$ with

$$\tau_\mu(P) = A_\mu(f).$$

THEOREM 2.3. *Let $(X, \mathbb{R})$ be a flow on a separable compact metric space with a $\mathbb{R}$-invariant Borel probability measure on X. If $f: X \to S^1$ is a function continuously differentiable with respect to the flow, such that $(1/2\pi i) f'(x)/f(x) > 0$ $\forall x \in X$, and if P_{K_f} is the projection assocciated to the cross section $K_f = \{x \in X \mid f(x) = 1\}$ by Lemma 1.2, then*

$$\tau_\mu(P_{K_f}) = \frac{1}{2\pi i} \int f'(x) \overline{f(x)}\, d\mu. \tag{7}$$

Proof. From Schwartzman's result we see that $K_f = \{x \in X \mid f(x) = 1\}$ is transversal to the flow and hence there exists an action of $\mathbb{Z}$ on K_f and a positive function $r_f: K_f \to S^1$, defined by

$$r_f(k) = \inf_{t > 0} \{t \mid kt \in K_f\},$$

such that $(X, \mathbb{R})$ is the flow built under the function r_f on K_f. As in Proposition 1.4, the trace τ_μ on $C^*(X, \mathbb{R})$ induces to give a trace $\operatorname{Ind}_{\bar{C}}(\tau_\mu)$ on $C^*(K_f, \mathbb{Z})$. A calculation shows that $\operatorname{Ind}_{\bar{C}}(\tau_\mu)$ corresponds to a $\mathbb{Z}$-invariant trace on $C^*(K_f)$, i.e., a finite positive Borel measure on K_f (not necessarily normalized), which we denote by ν. By Proposition 1.4, we see that

$$\tau_\mu = \operatorname{Ind}_{\bar{C}}(\operatorname{Ind}_{\bar{C}} \tau_\mu) = \operatorname{Ind}_{\bar{C}}(\tau_\nu)$$

and hence

$$\int_X h(x)\, d\mu = \int_{K_f} \int_0^{r_f(y)} h(\pi(y, r))\, dr\, d\nu$$

for any bounded measurable function $h: X \to C$, where $\Pi: K_f \times \mathbb{R} \to X$ is projection. We then note that $y \in K_f$,

$$\frac{1}{2\pi i} \int_0^{r_f(y)} f'(yr) \overline{f(yr)}\, dr = 1.$$

This follows from using the definition of $r_f(y)$ and from noting that

$$\int_0^{r_f(y)} f'(yr) \overline{f(yr)}\, dr = \log(f(yr)) \Big|_0^{r_f(y)}$$

for an appropriate branch of log. Therefore

$$\frac{1}{2\pi i} \int_X f'(x) \overline{f(x)}\, d\mu = \frac{1}{2\pi i} \int_{K_f} \int_0^{r_f(y)} f'(\pi(y, r)) \overline{f(\pi(y, r))}\, dr\, d\nu$$

$$= \int_{K_f} 1\, d\nu(y) = \nu(K_f),$$

$$\tau_\mu(P_{K_f}) = \tau_\mu(\langle m_{K_f}, m_{K_f} \rangle_{C^*(X, \mathbb{R})}),$$

where $m_{K_f}: K_f \times \mathbb{R} \to C$ is chosen as in the proof of Lemma 1.2.

We thus obtain

$$\begin{aligned} \tau_\mu(P_{K_f}) &= \tau_\mu(\langle m_{K_f}, m_{K_f} \rangle_{C^*(X, \mathbb{R})}) \\ &= \tau_\nu(\langle m_{K_f}, m_{K_f} \rangle_{C^*(K_f, \mathbb{Z})}) \\ &= \tau_\nu(\mathrm{Id}_{C^*(K_f, \mathbb{Z})}) \\ &= \nu(K_f). \end{aligned}$$

We thus obtain $\tau_\mu(P_{K_f}) = (1/2\pi i) \int_X f'(x) \overline{f(x)}\, d\mu$, as desired. ∎

If the flow $(X, \mathbb{R})$ is what Schwartzman terms spectrally determinate [cf. 24, Sect. 6], (a category which includes any recurrent flow on an arcwise-connected continuum X), then every element of $A_\mu(H^1(X, \mathbb{Z})) \cap \mathbb{R}^+$ corresponds to a transversal K_f. In such a situation the trace of the positive cone of $K_0(C^*(X, \mathbb{R}))$ (the collection of traces of actual projections in $M_n(C^*(X, \mathbb{R}))$ $\forall n \in \mathbb{Z}^+$), which is included in $A_\mu(H^1(X, \mathbb{Z}))$, is in fact equal to $A_\mu(H^1(X, \mathbb{Z})) \cap \mathbb{R}^+$. In other words to every element r of $A_\mu(H^1(X, \mathbb{Z})) \cap \mathbb{R}^+$ we can find a projection in $M_n(C^*(X, \mathbb{R}))$ for some positive integer n whose trace is equal to r (In fact $n = 1$ suffices $\forall r \in A_\mu(H^1(X, \mathbb{Z})) \cap \mathbb{R}^+$).

Remark 2.4. We note that Theorem 2.1 implies that for any continuous eigenfunction f_α of X with eigencharacter $\alpha > 0$,

$$\tau_\mu(P_{K_{f_\alpha}}) = \frac{1}{2\pi i} \int_X f'_\alpha(x) \overline{f_\alpha(x)}\, d\mu = \frac{1}{2\pi i} \int_X 2\pi i\alpha\, d\mu = \alpha.$$

A similar situation occurs in the following theorem, which was first stated and proved in [18] in a different fashion.

THEOREM 2.5 [18]. *Let $(Y, \mathbb{Z})$ be a minimal action of $\mathbb{Z}$ on the compact metric space with invariant measure ν, and suppose that $(Y, \mathbb{Z})$ has topologically discrete spectrum (i.e., $C^*(Y)$ is spanned by eigenfunctions). Then $\tau_\nu(K_0(C^*(Y, \mathbb{Z}))$ is equal to the inverse image in $\mathbb{R}$ of the eigenvalues for $(Y, \mathbb{Z})$ under the natural projection $\phi\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$.*

Proof. The fact that (Y, ν) has the structure of a compact abelian metric group with Haar measure and that there exists a monomorphism $\rho\colon \mathbb{Z} \to Y$ with dense range such that $T^n(y) = \rho(n)\, y$ is a standard result in topological dynamics. [cf. 26]. We form the induced flow $(X, \mathbb{R})$ where X is the quotient space $Y \times \mathbb{R}$ modulo the relation $(y, r) \sim (y\,\rho(n), r - n)$, $\forall n \in \mathbb{Z}$, $\forall (y, r) \in Y \times \mathbb{R}$. We again denote by π the projection of $Y \times \mathbb{R} \to X$. The space $Y \times \mathbb{R}$ has the natural structure of the product group, and $H = \{(\rho(n), -n) \in Y \times \mathbb{R} \mid n \in \mathbb{Z}\}$ is a closed subgroup of $Y \times \mathbb{R}$. The projection $\psi\colon Y \times \mathbb{R} \to Y \times \mathbb{R}/H$ has the same fibers as does π, and since π and ψ are both quotient maps we see that $Y \times \mathbb{R}/H$ is homeomorphic to X. Hence X has the structure of a compact abelian group. This implies that $H^1(X, \mathbb{Z})$ can be identified with the discrete group $\hat{X}$, [14], and by the Pontryagin duality $\hat{X}$ can be identified with $H^\perp = \{(\gamma, r) \in \widehat{(Y \times \mathbb{R})} \mid \gamma(\rho(n))\, e^{2\pi i r n} = 1,$ $\forall n \in \mathbb{Z}\}$. Hence using Theorem 2.1,

$$\tau_\nu(K_0(C^*(Y, \mathbb{Z}))) = \tau_\mu(K_0(C^*(X, \mathbb{R}))) = A_\mu(H^1(X, \mathbb{Z})),$$

which is seen to be $\{(1/2\pi i)\int_X (\gamma, r)'(x)\,\overline{(\gamma, r)}(x)\,d\mu \mid (\gamma, r) \in \hat{X} = H^\perp \subset Y\}$. By Remark 2.2, we note that for $(\gamma, r) \in \hat{X}$,

$$(\gamma, r)' \circ \pi(y, r) = 2\pi i r e^{2\pi i r t} \gamma(y)$$

so that

$$\begin{aligned} &\frac{1}{2\pi i}\int_X (\gamma, r)'(x)\overline{(\gamma, r)}(x)\,d\mu \\ &\quad= \frac{1}{2\pi i}\int_Y \int_0^1 2\pi i r e^{2\pi i r t}\gamma(y)\,\overline{e^{2\pi i r t}\gamma(y)}\,dt\,dv \\ &\quad= \frac{1}{2\pi i}\int_Y \int_0^1 2\pi i r\,dt\,dv \\ &\quad= r. \end{aligned}$$

Since $\{r \mid (\gamma, r) \in \hat{X}$ for some $\gamma \in \hat{Y}\} = \phi^{-1}(\{\gamma(\rho(1)) \mid \gamma \in \hat{Y}\})$, and since each $\gamma \in \hat{Y}$ is an eigenfunction for $(Y, \mathbb{Z})$ with eigenvalue $\gamma(\rho(1))$, we obtain the desired result. ∎

Remark 2.6. To further relate Theorem 2.5 to Remark 2.4 we note that each eigenfunction and eigenvalue for $(Y, \mathbb{Z})$ induces to give eigenfunctions and eigencharacters for the induced flow $(X, \mathbb{R})$, which also has topologically discrete spectrum. For each $\alpha > 0$ with $e^{2\pi i \alpha}$ an eigenvalue of the system $(Y, \mathbb{Z})$, there exists a $\mathbb{Z}$-space $(Y_\alpha, \alpha\mathbb{Z})$ which has topologically discrete spectrum and satisfies $(\mathrm{Ind}_{\alpha\mathbb{Z}}^{\mathbb{R}}(Y_\alpha), \mathbb{R}) = (X, \mathbb{R}) = (\mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}}(Y), \mathbb{R})$. The C^*-algebra $C^*(Y_\alpha, \alpha\mathbb{Z})$ is not necessarily *-isomorphic to $C^*(Y, \mathbb{Z})$ but is clearly strongly Morita equivalent to $C(Y, \mathbb{Z})$, the equivalence being given by

$$\begin{array}{ccccc} P_{K_{f_1}} C^*(X, \mathbb{R}) P_{K_{f_1}} & \text{——} & P_{K_{f_1}} C^*(X, \mathbb{R}) P_{K_{f_\alpha}} & \text{——} & P_{K_{f_\alpha}} C^*(X, \mathbb{R}) P_{K_{f_\alpha}} \\ \| & & & & \| \\ C^*(Y, \mathbb{Z}) & & & & C^*(Y_\alpha, \alpha\mathbb{Z}) \end{array}$$

Where f_1 and f_α are eigenfunctions with eigencharacters 1 and α, respectively, and K_{f_1}, K_{f_α} the corresponding cross-sections in X. A projection in $M_n(C(Y, \mathbb{Z}))$ of trace α for some n is found as in [19, Proposition 2.12].

3. Affine Transformations with Quasi-Discrete Spectrum

In this section we apply the formulas of the previous sections to calculate $(C^*(T^n, \mathbb{Z}))$, where $(T^n, \mathbb{Z})$ is an action of the integers on the n-torus by an affine transformation which is ergodic with respect to Haar measure and has quasi-discrete spectrum. The corresponding C^*-algebras may be viewed in several different lights. They are the simplest generalizations of the irrational rotation algebras, and they correspond to projective representations of certain countable discrete nilpotent groups. These two facts give one two different approaches to the use of the Pimsner–Voiculescu exact sequence. However, in what follows we shall be more interested in the induced flows for $(T^n, \mathbb{Z})$, which turn out to be of the form $(\Gamma\backslash N, \exp tX)$ where N is a simply connected nilpotent Lie group, Γ is a cocompact subgroup, and $X \in n$. (In this connection one could follow [7] and term the corresponding $C^*(T^n, \mathbb{Z})$ functions on noncommutative nilmanifolds). Again this approach is of most interest to us since we are then able to use Connes' theory to study certain non-unital crossed-product-by-$\mathbb{R}$ C^*-algebras which are strongly Morita equivalent to the unital $C^*(T^n, \mathbb{Z})$.

We begin by recalling that a dynamical system (X, T), where X is a compact Hausdorff space and T is a homeomorphism of X, is said to have *topologically quasi-discrete spectrum* if the algebra generated by its quasi-eigenfunctions is dense in $C^*(X)$. Here quasi-eigenfunctions are defined as follows: set G_1 equal to the set of topological eigenvalues for (X, T), and define G_{i+1} $i \in \mathbb{N}$, inductively, by setting G_{i+1} equal to $f \in C(X)$ such that $|f| \equiv 1$ and for which there exists a $g \in G_i$ with $f(Tx) = g(x) f(x)$. The set of quasi-eigenfunctions is defined by $G = \bigcup_{n=1}^{\infty} G_n$ In [30, Corollary, p. 318] it is shown that if (X, T) is a minimal dynamical system with quasi-discrete spectrum and no eigenvalues of finite order, then (X, T) is topologically conjugate to (K, A) where K is a compact abelian group and A is an affine transformation of K satisfying certain conditions. The system (K, A) is uniquely ergodic, distal and totally minimal. Many dynamical systems with quasi-discrete spectrum can be distinct from one another yet have the same group of eigenvalues; e.g., let

$$T^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n\text{-times}} \qquad \text{with} \qquad \theta_n = (z_1, z_2, \ldots, z_n) = (\lambda z_1, z_1 z_2, \ldots, z_{n-1} z_n)$$

where $\lambda = e^{2\pi i\alpha}\alpha$ irrational. Each of the systems (T^n, θ_n) has topologically quasi-discrete spectrum, with eigenvalues equal to $\{\lambda^n | n \in \mathbb{Z}\}$, yet they are all distinct systems and only (T^1, θ_1) has topologically discrete spectrum. Conversely, any system with discrete spectrum is completely determined up to conjugacy by its group of eigenvalues (and by definition has quasi-discrete spectrum).

We will restrict ourselves to the case where the compact group K in question is the n-torus T^n. In this case the results of [30] show that any minimal transformation of T^n without eigenvalues of finite order having quasi-discrete spectrum is conjugate to a minimal affine action on an

n-torus having quasi-discrete spectrum. Let θ be an affine transformation of the n-torus T^n so that there exists $M \in GL(n, \mathbb{Z})$ and $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$ with

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{R_\xi \cdot M} & \mathbb{R}^n \\ \downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle \phi_n} \\ T^n = \mathbb{R}^n/\mathbb{Z}^n & \xrightarrow{\theta} & T^n \end{array}$$

where $\phi_n: \mathbb{R}^n \to T^n$ is the natural projection, and $R_\xi(\mathbf{X}) = \mathbf{X} + \xi$, $\mathbf{X} \in \mathbb{R}^n$. (Any automorphism of T^n followed by a rotation can be expressed in such a fashion.) We identify the notation (ξ, M) with θ. Set $Z_p(M) = \ker((M^t)^p - I)\,\mathbb{Z}^n$, $\forall p \in \mathbb{N}$ (this is the $K_p(M)$ of [12], notation we have changed for obvious reasons). Note that $Z_1(M)$ is contained in $Z_p(M)$ for every p. The following two conditions give necessary and sufficient conditions for ergodicity w.r.t. Haar measure, as shown in [12]:

(1) $Z_1(M) = Z_p(M)$, $\forall p$

and

(2) $\bar{\xi} = (\xi_i)$ is rationally independent over $Z_1(M)$, i.e., if $\bar{m} \in Z_1(M)$ is such that $\langle \bar{\xi}, \bar{m} \rangle$ is a rational number, then $\bar{m} = 0$.

We restrict our study to transformations which satisfy (1) and (2) together with the following conditions:

(3) $Z_1(M) \neq 0$,

(4) $M^t - I$ is nilpotent.

Under these conditions it has been shown in [12] that $(\bar{\xi}, M)$ is conjugate (in the group of affine transformations of T^n) to the transformation $(\bar{\xi}', M')$ where M' is an upper triangular matrix whose bottom right $(k \times k)$ minor is the $k \times k$ identity matrix, where $0 < k = \dim_Q \ker(M' - I) = \dim_Q \ker((M')^t - I)$, and where $\bar{\xi}' = (\underbrace{0, ..., 0}_{n-k}, \xi_1, ..., \xi_k)$. Condition (2) then implies that $\xi_1, ..., \xi_k$ are rationally independent in the usual sense. We call (ξ', M') a *standard form* for (ξ, M) and from this point will assume that (ξ, M) is given in standard form.

A transformation satisfying (1)–(4) is minimal, uniquely ergodic with respect to Haar measure, and has quasi-discrete spectrum. Conversely any minimal transformation of the n-torus which has topologically quasi-discrete spectrum is conjugate to an affine transformation which must satisfy (1) through (4). [30, Sect. 3; 26, p. 116, Example (iv)] and hence can be put into a standard form. The C^*-algebras corresponding to such action are therefore simple and have a unique normalized trace.

The following observation is a slight variation of a more general method due to Auslander [28], which in turn is related to work of Auslander and

Hahn [29]. We indicate its proof here since the constructions and notation involved are crucial in the calculation of the range of the asymptotic cycle for the induced flow.

PROPOSITION 3.1. *Let $\theta = (\xi, M)$ be an affine transformation of the n-torus satisfying* (1)–(4) *and suppose that* (ξ, M) *is in standard form with* $\dim_Q \ker(M - I) = k$. *Then the induced flow* $(X, \mathbb{R}) = (\mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}}(T^n), \mathbb{R})$ *is topologically of the form* $(\Gamma\backslash N, \exp tX)$ *where* N *is a simply connected nilpotent* (*real*) *Lie group of dimension* $n+1$, Γ *is a cocompact subgroup, and the* $\mathbb{R}$ *action is given by translation on the right by* $\exp tX$ *for some* $X \in \mathfrak{N}$, *the Lie algebra of* N.

Proof. We recall that $\mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}}(T^n)$ is the quotient of $T^n \times \mathbb{R}$ by the relation

$$(z.r-1) \sim (\theta(z), r), \qquad \forall z \in T^n, \forall r \in \mathbb{R}.$$

Since M is unipotent, in fact upper triangular, there exists a unique upper triangular matrix $\bar{X} \in M(n, \mathbb{R})$ such that $\exp \bar{X} = M$. We define N as the topological space $\mathbb{R}^{n+1}$ and impose a group structure by employing a semi-direct product (since M is unipotent N will be a nilpotent Lie group),

$$\begin{aligned} (\mathbf{x}_1, t_1)(\bar{x}_2, t_2) &= (\bar{x}_1 + \exp(t_1 \bar{X})\, \mathbf{x}_2, t_1 + t_2), \\ (\mathbf{x}_1, t_1)^{-1} &= (-\exp(-t_1 \bar{X})(\mathbf{x}_1), -t_1). \end{aligned} \tag{5}$$

We then denote that $\mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}}(T^n)$ is precisely the left quotient of N by the subgroup D generated by

$$\begin{aligned} \mathbf{e}_j &= (0, \quad \underset{j\text{th spot}}{, 1,} \quad 0) \qquad 1 \leqslant j \leqslant n, \\ \mathbf{e}_{n+1} &= (0, \quad \underset{n-k+1}{, \xi_1,} \xi_2, \ldots, \xi_k, 1). \end{aligned} \tag{6}$$

(Although for purposes of writing up we use row vectors, we think of the vectors as column vectors when calculating $M\mathbf{x}$.)

So far this reasoning is a variation of the argument used in [28, Lemma 9, Sect. 9]. We now use Malcev's results on discrete subgroups of nilpotent Lie groups to find an automorphism of N mapping D onto the integer lattice of N, Γ. Let $x_1(t), x_{n+1}(t)$ be the unique one parameter subgroups in N such that $x_j(1) = \mathbf{e}_j$, $1 \leqslant j \leqslant n+1$.

For $1 \leqslant j \leqslant n$,

$$x_j(t) = (0, \ldots, \underset{j\text{th spot}}{t}, \ldots, 0)$$

If we denote by $(\exp tB - I)/B$ the expression $\sum_{j=1}^{\infty} t^j B^{j-1}/j!$, we note that the sum is finite for B nilpotent and hence we can define the invertible matrix $(\exp t\bar{X} - I)/\bar{X}$. With this notation,

$$x_{n+1}(t) = \left(\left(\frac{\exp t\bar{X} - I}{\bar{X}} \right) \left(\frac{\exp \bar{X} - I}{\bar{X}} \right)^{-1} \xi, t \right). \tag{7}$$

It is easy to verify that

(a) D is a cocompact subgroup of N (D is closed and $D\backslash N$ is compact).

(b) each element of $\mathbb{R}^n \times \mathbb{R} = N$ can be represented in the form

$$\prod_{i=1}^{n+1} x_i(t_i) \qquad \text{for some choice } (t_i) \in \mathbb{R}^{n+1}.$$

(c) For every integer $1 \leqslant j \leqslant n+1$ the collection of elements of the form $t_i \in \mathbb{R}$ form a closed normal subgroup of N, with

$$N_n \backslash N_{j+1} = \mathbb{R}, \qquad 1 \leqslant j \leqslant n.$$

(Hence $N_j \backslash N = \mathbb{R}$).

In other words, following the notation of [13], $x_1(1), x_2(1),..., x_{n+1}(1)$ form a system of *canonical coordinates of the second kind.* If we set

$$\bar{\mathbf{x}}_j(t) = x_j(t), \qquad 1 \leqslant j \leqslant n,$$
$$\bar{\mathbf{x}}_{j+1}(t) = (0,..., 0, t)$$

then the $\{\bar{\mathbf{x}}_j(1) | 1 \leqslant j \leqslant n+1\}$ also form a system of canonical coordinates of the second kind for N. Furthermore, the association $H: x_j(1) \to \bar{\mathbf{x}}_j(1)$ given an isomorphism of the group D and the group Γ generated by $\{\bar{\mathbf{x}}_j(1) | 1 \leqslant j \leqslant n+1\}$. Thus using Malcev's well-known results on cocompact subgroups of nilpotent Lei groups [16], H can be extended to give a group automorphism of N onto itself and thus a homeomorphism

$$\bar{H}: \quad D\backslash N \to \Gamma\backslash N.$$

We note that under the mapping H, the one parameter subgroup $\{(0,..., 0, t) | t \in \mathbb{R}\}$ is mapped onto the one-parameter subgroup

$$\left\{ -\left(\frac{\exp t\bar{X} - I}{\bar{X}} \right) \left(\frac{\exp \bar{X} - I}{\bar{X}} \right)^{-1} \xi, t \right), t \in \mathbb{R} \right\}, \tag{8}$$

X as in (7), which we shall henceforth denote by

$$\{\eta(t) | t \in \mathbb{R}\}.$$

Thus the induced flow corresponding to the given affine transformation of the n-torus is topologically equivalent to a one-parameter flow on a nilmanifold of the form $(\Gamma\backslash N, \eta(t))$ where $\mathbb{N}$ is a simply-connected nilpotent Lie group homeomorphic to $\mathbb{R}^{n+1}$ and Γ consists of the integer lattice of that group. Since under these conditions $\exp\colon \mathfrak{N} \to N$ is a diffeomorphism, we know there exists $X \in \mathfrak{N}$ with $\eta(t) = \exp tX$, and the proof of the proposition is complete. ▮

Remark 3.2. For further reference we comment on the calculation of X. We can identify N with the subgroup of $GL(n+1, \mathbb{R})$ of the form

$$\left\{\begin{pmatrix} M_1(t), & \bar{v} \\ 0 \cdots 0, & 1 \end{pmatrix} \bar{v} \in \mathbb{R}^n,\ t \in \mathbb{R},\ M_1(t) = \exp t\, X\right\}.$$

With respect to this identification, the element $X \in \mathfrak{n}$ can be expressed as an element of $M(n+1, \mathbb{R})$; thus we can identify X with the $(n+1) \times (n+1)$ dimensional matrix

$$X = \begin{pmatrix} \bar{X} & \left(\dfrac{\exp \bar{X} - I}{\bar{X}}\right) \bar{\xi} \\ 0, \ldots, 0, 0, & 0 \end{pmatrix}.$$

We note that for future reference that $M_1(t)$ must have its lower right $k \times k$ corner equal to the identity matrix, $\forall t \in \mathbb{R}$, since the lower right $k \times k$ corner of $\bar{\mathbf{X}}$ is the zero matrix.

We proceed now to the calculation of the trace of τ_μ on $K_0(C^*(X, \mathbb{R}))$ which in turn is equal to $\tau_\nu^*(K_0(C^*(T^n, \mathbb{Z})))$.

THEOREM 3.3. *Let $\theta\colon T^n \to T^n$ be a minimal transformation of the n-torus which has topologically quasi-discrete spectrum. Then*

$$\tau_\nu^*(K_0(C^*(T^n, \mathbb{Z})_\theta)) = \phi^{-1}(E),$$

where $E \to S^1$ is the set of topological eigenvalues for $(T^n, \mathbb{Z})_\theta$ and $\phi\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$ is the natural projection.

Proof. By Proposition 1.4 if suffices to calculate $\tau_\mu(K_0(C^*(X) \times \mathbb{R})) = A_\mu(H^1(X, \mathbb{Z}))$ where $(X, \mathbb{R})$ is as in Proposition 3.1. We recall that since N is simply connected, $H_1(\Gamma\backslash N, \mathbb{Z}) = \Pi_1(\Gamma\backslash N)/[\Pi_1(\Gamma\backslash N), \Pi_1(\Gamma\backslash N)]$ where $\Pi_1(X)$ represents the fundamental group of X, and hence

$$H_1(\Gamma\backslash N, \mathbb{Z}) = \Gamma/[\Gamma, \Gamma].$$

For any topological space X, there is a decomposition

$$H_1(X, \mathbb{Z}) = F_1(X, \mathbb{Z}) \oplus T_1(X, \mathbb{Z}),$$

where $F_1(X, \mathbb{Z})$ is the free first Betti group of X and $T_1(X, \mathbb{Z})$ is the maximal torsion subgroup of $H_1(X, \mathbb{Z})$. Since we are only concerned with $H^1(X, \mathbb{Z}) = (F_1(X, \mathbb{Z}))^*$, to calculate $H^1(\Gamma \backslash N, \mathbb{Z})$ we are only concerned with finding the free portion of $\Gamma/[\Gamma, \Gamma]$. To do so we note that $\Pi_1(\Gamma \backslash N) = \Gamma$ is generated by the elements

$$\{\mathbf{e}_j' = (0, \ldots, \underset{j\text{th spot}}{1}, \ldots, 0) \mid 1 \leqslant j \leqslant n+1\}.$$

Any element of Γ can therefore be written as $(\bar{v}, n)$, where $\bar{v} \in \mathbb{Z}^n$, $n \in \mathbb{Z}$. Thus a commutator in $[\Gamma, \Gamma]$ takes the form

$$(\bar{v}_1, n_1)(\bar{v}_2, n_2)(\bar{v}_1, n_1)^{-1}(\bar{v}_2, n_2)$$

$$(\bar{v}_1, n_1)(\bar{v}_2, n_2)(M^{-n_1}(-\bar{v}_1), -n_1)(M^{-n_2}(-\bar{v}_2), -n_2)$$

(where M is as in Proposition 3.1)

$$= (\bar{v}_1 + M^{n_1}(\bar{v}_2) + M_1^{n_2}(-\bar{v}_1) - \bar{v}_2, 0)$$

$$= [(M^{n_1} - I)\,\bar{v}_2 + (I - M^{n_2})\,\bar{v}_1, 0].$$

Then the dimension of $[\Gamma, \Gamma]$ (which is abelian so can be viewed as a $\mathbb{Z}$-submodule of $\mathbb{Z}^n \circledS \{0\} \subseteq \Gamma$) is equal to the dimension of the range of $\mathbb{Z}^n$ under the matrix $(M - I)$. Now M is an upper triangular matrix with ones along the diagonal, whose bottom right $k \times k$ minor is the $k \times k$ identity matrix, where $k = \dim_Q \ker(M - I)$, so that the rank of $M - I$ is $n - k$. Hence

$$\Gamma/[\Gamma, \Gamma] = \frac{\overbrace{\mathbb{Z} \oplus \mathbb{Z} \qquad \oplus \mathbb{Z} \circledS \mathbb{Z}}^{n+1 \text{ times}}}{(M - I)\,\mathbb{Z}^n}$$

$$= \frac{\overbrace{\mathbb{Z} \oplus \oplus \cdots \oplus \mathbb{Z}}^{n-k \text{ times}} \oplus \overbrace{\mathbb{Z} \oplus \cdots \mathbb{Z}}^{k \text{ times}} \oplus \cdots \overbrace{\mathbb{Z}}^{1 \text{ time}}}{\mathbb{Z} \text{ submodule of } \dim_Q = n - k}$$

$$= \qquad T_1 \oplus \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{k+1 \text{ times}},$$

where $T_1 = T_1(\Gamma \backslash N, \mathbb{Z})$ is the finite torsion subgroup of $H_1(\Gamma \backslash N, \mathbb{Z})$. Thus we have shown that the first Betti number of $\Gamma \backslash N$ is equal to $k + 1$, so that the dimension of $F_1(\Gamma\backslash N)$ as a $\mathbb{Z}$ module is $k+1$. If we denote by π the projection $\pi\colon N \to \Gamma\backslash N$ and by ψ the surjection

$$\psi\colon \Gamma = \pi_1(\Gamma\backslash N) \to \Gamma/[\Gamma, \Gamma] = H_1(\Gamma\backslash N, \mathbb{Z}),$$

then $F_1(\Gamma\backslash N, \mathbb{Z})$ is generated by the cycles

$$\psi([\pi \circ \gamma_j]), \qquad 1 \leqslant j \leqslant k+1,$$

where each γ_j is a curve $\gamma_j\colon [0, 1] \to N$, with

$$\gamma_j(t) = (0,..., 0, \underset{n-k+j}{t},..., 0), \qquad 1 \leqslant j \leqslant k+1$$

so that $\pi \circ \gamma_j$ is a loop in $\Gamma\backslash N$, with $[\pi \circ \gamma_j] \in \pi_1(\Gamma\backslash N) = \Gamma$.

There are now two slightly different approaches to calculating $\tau_\mu^*(K_0(C^*(\Gamma\backslash N, \mathbb{R}))) = \tau_\nu^*(C^*(T^n, \mathbb{Z}))$. One can emphasize the manifold structure by finding a dual basis to the $\psi([\pi \circ \gamma_j])$ in terms of one-forms, and thus calculate $H^1(\Gamma\backslash N, \mathbb{Z})$ by finding the integral one-form representatives of $H^1(\Gamma\backslash N, \mathbb{R})_{\text{de Rham}}$. If this direction is taken, then a basis dual to the basis given for $F_1(\Gamma\backslash N, \mathbb{Z})$ is given by the cocycles obtained from the $n+1$ integral one forms

$$\omega_j \circ \pi(t_1,..., t_{n+1}) = \frac{1}{2\pi i} \exp(2\pi i\, t_{n-k+j})\, d(\exp 2\pi i\, t_{n-k+j}), \qquad 1 \leqslant j \leqslant k+1.$$

The cycles $[\omega_j \circ \pi]$ can be regarded as sitting inside $H^1(\Gamma\backslash N, \mathbb{R})_{\text{de Rham}}$.

From this point of view the next step is to examine the image of the Ruelle–Sullivan current which following [4] we denote by C. We recall that if the flow $(\Gamma\backslash N, \mathbb{R})$ is expressed in terms of the vector field X on $\Gamma\backslash N$, then

$$\langle [\omega], C \rangle = \int_{\Gamma\backslash N} \omega(X)\, d\mu.$$

If instead the algebraic–topological point of view is used, one examines the Cech cohomology $G(\Gamma\backslash N)/E(\Gamma\backslash N) = H^1(\Gamma\backslash N, \mathbb{Z})$. In this case the $[\omega_j]$ corresponds to $[f_j] \in G(\Gamma\backslash N)/E(\Gamma\backslash N)$ where

$$f_j \circ \pi(t_1,..., t_{n+1}) = \exp(2\pi i t_{n-k+j}), \qquad 1 \leqslant j \leqslant k+1$$

and the image of the Schwartzman asymptotic cycle is given by

$$\langle f_j, A_\mu \rangle = \frac{1}{2\pi i} \int_{\Gamma\backslash N} \bar{f}_j f_j'\, d\mu = \langle [\omega_j], C \rangle, \qquad 1 \leqslant j \leqslant k+1.$$

We now note that each f_j is a continuous eigenfunction to the flow $(\Gamma\backslash N, \mathbb{R})$, with eigencharacter $-\xi_j$ (where we define $-\xi_{k+1}=1$). This follows from the fact that the lower right $k\times k$ corner of $\exp(tX)$ is the identity matrix for every t, so that

$$\begin{aligned}
&f_j\circ\pi((t_1,..., t_{n+1})\cdot\eta(t))\\
&\quad=f_j\circ\pi((t_1,..., t_{n+1})(*; \xi_1^t, -\xi_2^t,..., -\xi_k^t, t)\\
&\quad=f_j\circ\pi(*, t_{n-k+1}-\xi_1 t, t_{n-k+2}-\xi_2 t,..., t_{n+1}+t)\\
&\quad=\exp 2\pi i(t_{n-k+j}-\xi_j t), \qquad 1\leqslant j\leqslant k+1\\
&\quad=\exp(-\xi_j t) f_j\circ\pi((t_1,..., t_{n+1})).
\end{aligned}$$

Hence using the results of section 2, $\langle f_j, A_\mu\rangle=-\xi_j$, $1\leqslant j\leqslant k+1$. We note also that $\bar{f}_j$, $1\leqslant j\leqslant k$ and f_{k+1} satisfy the condition $(1/2\pi i)\bar{f}_j'\bar{\bar{f}}_j>0$ for $\xi_j>0$; therefore there exists cross sections $K_1,..., K_{k+1}$ and corresponding projections P_{K_j}, $1\leqslant j\leqslant k+1$, in $C^*(\Gamma\backslash N, \mathbb{R})$ associated to the $\bar{f}_j$ where

$$\begin{aligned}
\tau_\mu(P_{K_j})&=+\xi_j, \qquad 1\leqslant j\leqslant k,\\
\tau_\mu(P_{K_{k+1}})&=1.
\end{aligned}$$

Hence $\tau_\nu^*(K_0(C^*(T^n, \mathbb{Z})))=\tau_\mu^*(K_0(C^*(\Gamma\backslash N, \mathbb{R})))$ contains $\bigoplus_{j=1}^k \mathbb{Z}\xi_j\oplus\mathbb{Z}\subset\mathbb{R}_1$ and an application of Connes' Corollary 2 shows that in fact

$$\tau_\nu^*(K_0(C^*(T^n, \mathbb{Z})))=\bigoplus_{j=1}^{k}\mathbb{Z}\xi_j\oplus\mathbb{Z}.$$

Since $(\bigoplus_{j=1}^k\mathbb{Z}\xi_j\oplus\mathbb{Z})=\phi^{-1}(E(T^n, \mathbb{Z}))$ where $E(T^n, \mathbb{Z})$ represents the (continuous) eigenvalues of $(T^n, \mathbb{Z})$ and where $\phi\colon\mathbb{R}\to\mathbb{R}/\mathbb{Z}=S^1$, the proof of our theorem is complete. ∎

Remark 3.4. The referee has inquired as to whether Riedel's proof of Theorem 2.5 can be easily modified to prove our Theorem 3.3. A portion of Riedel's proof of this theorem [18, Lemma 3.3] is based on induction on the number of generators for the group of eigenvalues, and to prove the induction for $i=1$ uses the known fact that $K_0(A_\alpha)=\mathbb{Z}\oplus\mathbb{Z}\alpha=\tau^*(K_0(A_\alpha))$, where A_α is the irrational rotation algebra. In our case, we saw on p. 20 that infinitely many nonconjugate affine systems (T^n, θ_n) with quasi-discrete spectrum can have their eigenvalue groups all generated by the number $e^{2\pi i\alpha}$. Here the groups $K_0(C^*(T^n, \theta_n))$ and the ranges $\tau^*(K_0(C((T^n, \theta_n))$ are not accessible by a direct application of the Pimsner–Voiculescu–Rieffel results.

Remark 3.5. The Pimsner–Voiculescu exact sequence allows one to easily calculate the group structure of $K_*(C^*(T^n, \mathbb{Z}))$, for $C^*(T^n, \mathbb{Z})$

associated to any given affine transformation of T^n; one uses the fact that an affine transformation is homotopic to an automorphism and applies the isomorphism ch: $K_*(C(T^n)) \to H^*(T^n)$ given by the Chern character, as pointed out to the author by C. C. Moore.

4. Applications: The Discrete Heisenberg Group and Anzai's Skew Product on the Torus

We now restrict ourselves to the specific case $n=2$ and $\theta\colon T^2 \to T^2$ is given by $\theta((w, z)=(zw, \lambda z)$ where $\lambda = e^{2\pi i\alpha}$, for α irrational. Denote by H_α the C^*-algebra $C^*(T^2, \mathbb{Z})_\theta$ which is the crossed product of $C^*(T^2)$ by the action θ^* of $\mathbb{Z}$. Since θ is a minimal transformation, H_α is simple. The notation $\{H_\alpha\}$ is chosen for the reason that H_α is generated by the three elements U, V, W satisfying the relations

$$UV = \lambda VU,$$

$$UW = VWU,$$

$$VW = WV,$$

i.e., H_α corresponds to a projective representation of the three-dimensional Heisenberg group. A straightforward application of the method discussed in Section 3 show that the flow induced from θ is given by right action of the one-parameter subgroup

$$\phi(t) = \left(\left(\frac{t}{2} - \frac{t^2}{2}\right)\alpha,\ -\alpha t,\ t\right)$$

on the homogeneous space $H_{\mathbb{Z}}\backslash H_{\mathbb{R}}$, where $H_{\mathbb{R}}$ is the three dimensional real Heisenberg group

$$(x, y, z), \qquad x, y, z \in \mathbb{R},$$

$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2 + z_1 y_2,\ y_1 + y_2,\ z_1 + z_2)$ and $H_{\mathbb{Z}}$ is the integral subgroup of $H_{\mathbb{R}}$. (An application of Remark 3.2 obtains the representation of $H_{\mathbb{R}}$ as upper triangular matrices in $M(3, \mathbb{R})$ with ones along the diagonal.)

A calculation of the K-groups of $C^*(T^2, \mathbb{Z})$ & $C^*(H_{\mathbb{Z}}\backslash H_{\mathbb{R}}, \mathbb{R})$ can be made by the Pimsner–Voiculescu exact sequence or Connes' Thom isomorphism theorem and yields

$$K_0(H_\alpha) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z},$$

$$K_1(H_\alpha) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

Using the methods of the previous section we see that $\tau_\nu^*(K_0(C^*(T^2, \mathbb{Z}))) = \mathbb{Z} \oplus \alpha\mathbb{Z}$. This last equality implies the first part of

THEOREM 4.1. *Let H_α and H_β be given as above where α and β are irrational numbers. Then H_α is *-isomorphic to H_β if and only if $\alpha \equiv \beta \bmod \mathbb{Z}$ or $\alpha \equiv 1-\beta \bmod \mathbb{Z}$. Furthermore H_α is strongly Morita equivalent to H_β if and only if there exists $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2, \mathbb{Z})$ with*

$$\alpha = \frac{a\beta + b}{c\beta + d}. \tag{1}$$

Proof. As in [19] since $(T^2, \mathbb{Z})_\alpha$ is uniquely ergodic, each H_α has a unique normalized trace; hence any isomorphic $H_\alpha \equiv H_\beta$ implies that

$$\begin{aligned} \tau_{\nu_\alpha}^*(K_0(H)) &= \tau_{\nu_\beta}^*(K_0(H)) \\ &\Rightarrow \alpha = \beta \text{ or } 1-\beta \bmod \mathbb{Z}. \end{aligned}$$

On the other hand if $\alpha \equiv 1-\beta \bmod \mathbb{Z}$ we note that θ_α is conjugate to $(\theta_\beta)^{-1}$ which is in turn conjugate to θ_β, so that $H_\alpha \cong H_\beta$.

The second part of the theorem is just a "lifting" of the results of Rieffel proved for irrational rotation algebras in [19] which also can be applied to our case. Thus if H_α and H_β are strongly Morita equivalent there exists $r \in \mathbb{R}$ with $r(\tau_{\nu_\alpha}^*(K_0(H_\alpha))) = \tau_{\nu_\beta}^*(K_0(H_\beta))$. The same source, Theorem 4 of [19], shows the existence of a $\gamma \in GL(2, \mathbb{Z})$ satisfying (1).

In the other direction we note that if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2, \mathbb{Z})$ is such that $\alpha = (-a\beta + b)/(-c\beta + d) \bmod \mathbb{Z}$ so that $\left(\begin{smallmatrix} -a & b \\ -c & d \end{smallmatrix}\right) \in GL(2, \mathbb{Z})$ satisfies (1), then, roughly speaking, the orbits of $\eta_\alpha(t)$ can be considered equivalent to those of $\eta_\beta(t)$.

For the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2, \mathbb{Z})$ induces an automorphism $\gamma^*: H_{\mathbb{Z}} \to H_{\mathbb{Z}}$ given by the correspondences

$$\begin{aligned} (0, 1, 0) &\to (0, a, c) \\ (0), 0, 1) &\to (0, b, d), \\ (1, 0, 0) &\to (\det \gamma, 0, 0). \end{aligned} \tag{2}$$

Once again, using the theorems of Malcev, γ^* may be extended to an automorphism

$$\gamma^*: H_{\mathbb{R}} \to H_{\mathbb{R}}$$

Indeed

$$\begin{aligned}
\gamma^*((t,0,0) &= (\det\gamma\cdot t, 0, 0),\\
\gamma^*((0,t,0)) &= \left(ac\,\frac{(t^2-t)}{2}, at, ct\right), \qquad (3)\\
\gamma^*((0,0,t)) &= \left(bd\,\frac{(t^2-t)}{2}, bt, dt\right).
\end{aligned}$$

We note

$$\gamma^*(\eta_\beta(t)) = \left(\det\gamma\cdot\beta\,\frac{(t-t^2)}{2} + ac\beta\,\frac{(\beta t^2+t)}{2} + \frac{bd}{2}(t^2-t) - cb\beta t^2,\right.$$
$$\left. -a\beta t + bt, (-c\beta+d)\,t\right) \in H_{\mathbb{R}}. \qquad (5)$$

Since $\gamma^*: H_{\mathbb{Z}} \to H_{\mathbb{Z}}$ it follows that we have a topological homeomorphism

$$\bar{\gamma}^*: H_{\mathbb{Z}}\backslash H_{\mathbb{R}} \to H_{\mathbb{Z}}\backslash H_{\mathbb{R}}.$$

Our aim now is to show that the flows generated by $\eta_\alpha(t)$ and $\eta_\beta(t)$ on $H_{\mathbb{Z}}\backslash H_{\mathbb{R}}$ (η_α, η_β as in Proposition 3.1) are orbit equivalent, i.e., to find a self-homeomorphism ϕ: $H_{\mathbb{Z}}\backslash H_{\mathbb{R}}$ carrying η_α orbits onto η_β orbits. In fact we calculate an element $w = (0, x, y) \in H_{\mathbb{R}}$ such that $\phi = R_w \bar{\gamma}^*$, where R_g denotes translation on the right by $g \in H_{\mathbb{R}}$ on $H_{\mathbb{Z}}\backslash H_{\mathbb{R}}$. Letting z be an arbitrary element of $H_{\mathbb{R}}$, and denoting by π the surjection $\pi: H_{\mathbb{R}} \to H_{\mathbb{Z}}\backslash H_{\mathbb{R}}$, then

$$\begin{aligned}
&R_w \circ \bar{\gamma}^* \circ R_{\eta_\beta(t)} \circ \bar{\gamma}^{*-1} \circ R_w^{-1}(\pi(z))\\
&\quad = R_w \circ \bar{\gamma}^* \circ R_{\eta_\beta(t)} \circ \bar{\gamma}^{*-1}(\pi(z\,w^{-1}))\\
&\quad = R_w \circ \bar{\gamma}^* \circ R_{\eta_\beta(t)}(\pi(\gamma^{*-1}(z)\,\gamma^{*-1}(w^{-1})))\\
&\quad = R_w \circ \bar{\gamma}^* \circ \pi(\gamma^{*-1}(z)\,\gamma^{*-1}(w^{-1})\,\eta_\beta(t))\\
&\quad = R_w \pi(z\,w^{-1}\gamma^*(\eta_\beta(t)))\\
&\quad = \pi(z\,w^{-1}\gamma^*(\eta_\beta(t))\,w)\\
&\quad = R_{w^{-1}\gamma^*(\eta_\beta(t))\,w}(\pi(z)).
\end{aligned}$$

Thus if we can find a constant $k \in \mathbb{R}$ and $w \in H_{\mathbb{R}}$ with $w^{-1}\gamma^*(\eta_\beta(t))\,w = \eta_\alpha(kt)$ (5), the map $\phi = R_w \circ \bar{\gamma}^*: H_{\mathbb{Z}}\backslash H_{\mathbb{R}} \to H_{\mathbb{Z}}\backslash H_{\mathbb{R}}$ will carry orbits of the flow $(H_{\mathbb{Z}}\backslash H_{\mathbb{R}}, \eta_\beta(t))$ onto the orbits of $(H_{\mathbb{Z}}\backslash H_{\mathbb{R}}, \eta_\alpha(t))$ as desired. Taking $k=(-c\beta+d)$, Eq. (5) implies that we must have

$$(0, x, y)\,\gamma^*(\eta_\beta(t))(xy, -x, -y)=\eta_\alpha((-c\beta+d)\,t). \tag{6}$$

Using the fact that $\alpha=(-a\beta+b)/(-c\beta+d)$, and by combining Eq. (4) with (6), it is possible to solve for $(x, y)\in\mathbb{R}^2$ that satisfies (6) as $ad-bc=\pm 1\neq 0$.

Hence $C^*(H_{\mathbb{Z}}\backslash H_{\mathbb{R}}, \mathbb{R})_{\eta_\beta}$ is *-isomorphic to $C^*(H_{\mathbb{Z}}\backslash H_{\mathbb{R}}, \mathbb{R})_{\eta_\alpha}$ which implies that $C^*(T^2, \mathbb{Z})_\alpha$ is strongly Morita equivalent to $C^*(T^2, \mathbb{Z})_\beta$ as desired. ∎

Remark 4.2. An important factor in our ability to set up a Morita equivalence was the fact that

$$n-k=2-1=1, \qquad n, k \text{ as in Proposition 3.1.}$$

Thus if one wishes to obtain an extension of Theorem 4.1 to the more general situation of Sect. 3, one can show that a necessary condition for the s. Morita equivalence of $C^*(T^n, \mathbb{Z})_{M_1,\xi_1}\to C^*(T^n, \mathbb{Z})_{M_2,\xi_2}$, where M_1 and $M_2\in GL(n, \mathbb{Z})$ satisfy the conditions of Proposition 3.1 is the existence of a $k\in\mathbb{N}$, $k=k_1=k_2$ an $A\in GL(k+1, \mathbb{Z})$ and $r\in\mathbb{R}$ such that

$$A\begin{pmatrix}\xi_{1,1}\\ \vdots\\ \xi_{1,k}\\ 1\end{pmatrix}=r\begin{pmatrix}\xi_{2,1}\\ \vdots\\ \xi_{2,k}\\ 1\end{pmatrix}.$$

However, for cases where $n-k>1$, even when $M_1=M_2$ this condition is not necessarily sufficient to construct a automorphism $\bar{\gamma}^*\, N^1_{\mathbb{R}}\to N^2_{\mathbb{R}}$ with the desired properties.

APPENDIX[1]

In the above paper the author indicated a method for the calculation of the range of the trace on the K_0-group of certain transformation group C^*-

[1] After writing this paper we received a preprint of M. Pimsner's "Range of traces on K_0 of reduced crossed products by free groups," in which he proves a very general result in a different fashion which allows him to give another proof of Riedel's theorem (Theorem 2.5) and which can be used to prove our Theorem 3.3. We comment on the relationship between our work and Pimsner's in the Appendix.

algebras, and used this method to calculate the range of the trace of the K_0-groups of C^*-algebras generated by tranformations with quasi-discrete spectrum. W. Paschke has recently brought to our attention a preprint by Pimsner [33], in which a general formula for the range of the trace on the K_0-group of the reduced crossed product (by a free group) C^*-algebra is given. Here the trace comes from a group-invariant trace on the original algebra. When $\mathbb{Z}$ acts on a compact metric space, the formula considerably facilitates the calculation of the tracial invariant in the transformation group $\mathbb{C}^*$-algebra. The aim of this note is to reproduce the formula for this case by means of a natural extension of the methods used above, and to discuss the significance of the formula in this context.

We keep the notation above and suppose that $\mathbb{Z}$ acts on the compact metric space Y via a homeomorphism $T: Y \to Y$. We may form $(X, \mathbb{R})$, the "mapping torus" for (Y, T), i.e., the quotient space of $(Y \times \mathbb{R})$ by the $\mathbb{Z}$ action $(y, r) \cdot n = (T^n y, r - n)$, and $\mathbb{R}$ acts on X by $(\Pi(x, r)) \cdot t = \Pi(x, r + t)$, where $\Pi: Y \times \mathbb{R} \to X$ is the covering projection. A $\mathbb{Z}$-invariant measure ν on Y induces to a $\mathbb{R}$-invariant measure μ on X.

Let $C^*(Y, \mathbb{Z})$, $C^*(X, \mathbb{R})$ denote the transformation group C^*-algebras with traces τ_ν and τ_μ, respectively. Results of Connes show that $\tau_\mu^*(K_0(C^*(X, \mathbb{R}))) = A_\mu(H^1(X, \mathbb{Z}))$, where A_μ is the asymptotic cycle of Schwartzman and $H^1(X, \mathbb{Z})$ is Cech cohomology. Since $C^*(Y, \mathbb{Z})$ is strongly Morita equivalent to $C^*(X, \mathbb{R})$ with the traces τ_ν and τ_μ corresponding to one another with respect to this equivalence (Proposition 1.4), it follows that

$$\tau_\nu^*(K_0(C^*(Y, \mathbb{Z}))) = \tau_\mu^*(K_0(C^*(X, \mathbb{R}))).$$

Thus to calculate the range of the trace τ_ν^* on $K_0(C^*(Y, \mathbb{Z}))$, it suffices to calculate the range of the asymptotic cycle A_u on $H^1(X, \mathbb{Z})$. This involves the calculation of $H^1(X, \mathbb{Z})$ from $H^1(Y, \mathbb{Z})$ and $H^0(Y, \mathbb{Z})$. This was done in the examples of section 3 by a somewhat lengthy process. There is, however, a much simpler method to calculate $H^1(X, \mathbb{Z})$ through an exact sequence which we indicate here. First, we fix our notation.

If M is a compact metric space, we denote by $G(M)$ the mulfiplicative group of continuous functions mapping M into S^1; $C_{\mathbb{R}}(M)$ the additive group of continuous real-valued functions on M, and $E(M)$ the subgroup of $G(M)$ consisting of those functions which can be expressed as $\exp(2\pi i\, h(m))$ for $h \in C_{\mathbb{R}}(M)$. We regard $H^1(M, \mathbb{Z})$ as the quotient group $G(M)/E(M)$ and $H^0(M, \mathbb{Z})$ as the subgroup of $C_{\mathbb{R}}(M)$ consisting of those continuous functions on M which are integer-valued. If $\phi: M \to M$ is a homeomorphism it induces group endomorphisms of $G(M)$, $E(M)$, $C_{\mathbb{R}}(M)$, $H^1(M, \mathbb{Z})$, and $H^0(M, \mathbb{Z})$. If $f \in G(M) \cup C_{\mathbb{R}}(M)$ we denote the corresponding endomorphisms by $f \to f \circ \phi$; if $u \in H^1(M, \mathbb{Z}) \cup H^0(M, \mathbb{Z})$ we denote the endomorphisms by $u \to \phi^*(u)$. If $f \in G(M)$ let $[f]$ denote its equivalence class in $H^1(M, \mathbb{Z})$. Note $[f \circ \phi] = \phi^*([f])$.

With the above notation in hand, we can state the following lemma, an analogue of part of the Wang exact sequence which was used for manifolds in [32]:

LEMMA A. *Let Y be a compact metric space and $T\colon Y \to Y$ a homeomorphism, and let $(X, \mathbb{R})$ be the flow induced from $(Y, \mathbb{Z})$. Then there is an exact sequence*

$$H^0(Y, \mathbb{Z}) \xrightarrow{T^* - \mathrm{Id}} H^0(Y, \mathbb{Z}) \xrightarrow{\alpha} H^1(X, \mathbb{Z}) \xrightarrow{\beta} H^1(Y, \mathbb{Z}) \xrightarrow{T^* - \mathrm{Id}} H^1(Y, \mathbb{Z}).$$

Proof. The maps α and β are given by

$$\alpha(h) = [e_h], \qquad \text{where } e_h \circ \Pi(y, t) = \exp 2\pi i t h(y).$$

(Here $\Pi\colon Y \times \mathbb{R} \to X$ is the covering projection restricted to $Y \times [0, 1]$) and

$$\beta([f]) = [f_{Y,0}], \qquad \text{where } f_{Y,0}(y) = f \circ \Pi(y, 0).$$

The verification of exactness is routine. ∎

Thus to compute the range of A_μ on $H^1(X; \mathbb{Z})$ one need only use the maps given in Lemma A to compute representatives for $H^1(X, \mathbb{Z})$ in $G(X)$ and compute the range of A_μ by using the formula given in section 4 of [24]

$$A_\mu([f]) = \int_X \lim_{t \to \infty} \frac{1}{2\pi i t} \Delta_{(x, xt)} \arg f \, d\mu.$$

Here $\Delta_{(x, xt)} \arg f$ is the change in argument of f as x flows to $x \cdot t$; then $\lim_{t \to \infty} (1/2\pi i t) \Delta_{(x, xt)} \arg f$ exists for μ-almost all $x \in X$. By doing this we can obtain

THEOREM B. (*See* [33, *Corollary* 4]). *Let Y be a compact separable metric space and $T\colon Y \to Y$ a homeomorphism, and suppose that ν is a T-invariant measure on Y. Then*

$$\tau_\nu(K_0(C^*(Y, \mathbb{Z}))) = \left\{ \int_Y h \, d\nu, \ h \in \mathrm{Log}(Y, T) \right\},$$

where $\mathrm{Log}(Y, T)$ *is the subgroup of $C_{\mathbb{R}}(Y)$ consisting of functions h satisfying*

$$f(Ty) = e^{2\pi i h(y)} f(y) \qquad \text{for some } f \in G(Y).$$

Proof. By our remarks earlier, we need only prove that

$$A_\mu(H^1(X, \mathbb{Z})) = \left\{ \int_Y h\,d\nu,\ h \in \operatorname{Log}(Y, T) \right\},$$

where $(X, \mathbb{R})$ is the flow induced from $(Y, \mathbb{Z})$ and μ is the $\mathbb{R}$-invariant measure induced from ν. By the exact sequence of Lemma A, an element of $H^1(X, \mathbb{Z})$ can be represented by a function $g_{(m,f,l)} \in G(X)$, with $m \in H^0(Y, \mathbb{Z})$, $f \in G(Y)$, $l \in C_{\mathbb{R}}(Y)$ s.t. $m \in im(T^* - \mathrm{Id}) \subset H^0(Y, \mathbb{Z})$, $[f] \in \ker(T^* - \mathrm{Id}) \subset H^1(Y, \mathbb{Z})$, $f \circ T(y) = f(y)\, e^{2\pi i l(y)}$, where

$$g_{(m,f,l)} \circ \Pi(y, r) = e^{2\pi i m(y) r} f(y)\, e^{2\pi i l(y) r},$$

where again Π is restricted to $Y \times [0, 1]$. Hence we write $g \circ \Pi(y, r) = f(y)\, e^{2\pi i h(y) r}$, where

$$h(y) = l(y) + m(y) \in \operatorname{Log}(Y, T), \qquad [f] \in \ker(T^* - \mathrm{Id}).$$

To compute $A_\mu([g])$ we must compute $\lim_{t \to \infty} (1/2\pi t)\, \Delta_{(x, xt)} \arg g$ which exists for almost all x in X. Hence

$$\lim_{n \to \infty} \frac{1}{2\pi n} \Delta_{(x, x \cdot n)} \arg g, \qquad n \in \mathbb{N},$$

exists for almost all $x \in X$. Hence

$$\lim_{n \to \infty} \frac{1}{2\pi n} \Delta_{((y,r),(y,r) \cdot n)} \arg g \circ \Pi, \qquad n \in \mathbb{N},$$

exists for almost all $(y, r) \in Y \times [0, 1]$. But

$$\begin{aligned} &\Delta_{((y,r),(y,r) \cdot n)} \arg g \circ \Pi \\ &\quad = 2\pi((1 - r)\, h(y) + h(Ty) + h(T^2 y) + \cdots + h(T^{n-1} y) + r h(T^n(y))) \end{aligned}$$

so that

$$\begin{aligned} &\frac{1}{2\pi n} \Delta_{((y,r),(y,r) \cdot n)} \arg g \circ \Pi \\ &\quad = \frac{((1 - r)\, h(y) + h(T(y)) + h(T^2(y)) + \cdots + h(T^{n-1} y) + r h(T^n(y)))}{n} \\ &\quad = \frac{h(y) + h(T(y)) + \cdots + h(T^{n-1}(y)) - r(h(y) - h(T^n(y)))}{n}. \end{aligned}$$

Hence

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{2\pi n} {}_{((y,r),(y,r)\cdot n)} \arg g\circ \Pi \\
&= \lim_{n\to\infty} \left(\frac{\sum_{i=0}^{n-1} h(T^i(y))}{n} - \frac{r(h(y)-h(T^n(y)))}{n} \right) \\
&= \lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} h(T^i(y))}{n}.
\end{aligned}$$

But by the ergodic theorem [31], since $h\in C_{\mathbb{R}}(Y)\subset L^1(Y,\nu)$, the limit $\lim_{n\to\infty}(1/n)\sum_{i=0}^{n-1} h(T^i(y))$ exists for ν-almost every $y\in Y$ and defines a function $\bar{h}$ in $L^1(Y)$, with

$$\int_Y \bar{h}(y)\,d\nu = \int_Y h(y)\,d\nu.$$

Hence

$$\begin{aligned}
&A_\mu([g]) \\
&= \int_X \lim_{t\to\infty} \frac{1}{2\pi T} \Delta_{(x,xt)} \arg g\, d\mu \\
&= \int_Y \int_0^1 \lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} h(T^i(y))}{n}\, dm\, d\nu \\
&= \int_Y \int_0^1 \bar{h}(y)\, dm\, d\nu \\
&= \int_Y \bar{h}(y)\, d\nu \\
&= \int_Y h(y)\, d\nu.
\end{aligned}$$

Hence $A_\mu(H^1(X,\mathbb{Z}))\subset\{\int_Y h(y)\,d\nu,\ h\in \mathrm{Log}(Y),T)\}$. To show the reverse inclusion we note that if $h\in\mathrm{Log}(Y,T)$, then for $f\in G(Y)$ with $f\circ T(y)=f(y)\,e^{2\pi i h(y)}$, $g_{(f,h)}\in G(X)$ can be defined with

$$g_{(f,h)}\circ(y,r)=f(y)\,e^{2\pi i h(y)r}.$$

Then

$$A([g])=\int_Y h(y)\,d\nu$$

through the same argument as above. Hence

$$A_\mu(H^1(X, \mathbb{Z})) = \left\{ \int_Y h(y)\, d\nu,\ h \in \operatorname{Log}(Y, T) \right\}.$$

Remark. If $[f] \in \ker(T^* - \mathrm{Id}) \subset H^1(Y, \mathbb{Z})$, then $f(Ty) = f(y)\, e^{2\pi i h(y)}$ for some $h \in \operatorname{Log}(Y, T)$. If $f(y)\, e^{2\pi i h_1(y)} = f(y)\, e^{2\pi i h_2(y)} = f(Ty)$, we must have $h_1(y) - h_2(y) \in H^0(Y, \mathbb{Z})$. Hence the map A_μ: $H^1(X, \mathbb{Z}) \to \mathbb{R}$ projects via β: $H^1(X, \mathbb{Z}) \to H^1(Y, \mathbb{Z})$ to give a map $\tilde{A}_\nu$: $\ker(T^* - \mathrm{Id}) \to \mathbb{R}/\tau(H^0(Y))$,

$$\begin{array}{ccc} H^1(X, \mathbb{Z}) & \xrightarrow{A_\mu} & \mathbb{R} \\ \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \phi} \\ H^1(Y, \mathbb{Z}) \supset \ker(T^* - \mathrm{Id}) & \xrightarrow{\tilde{A}_\nu} & \mathbb{R}/\tau(H^0(Y, \mathbb{Z})). \end{array}$$

This map $\tilde{A}_\nu$ is the Δ_τ of Pimsner [34]. The range of the trace τ^* on $K_0(C^*(Y, \mathbb{Z}))$ is just $\phi^{-1}(\tilde{A}_\nu(\ker(T^* - \mathrm{Id})))$ where

$$\phi: \mathbb{R} \to \mathbb{R}/\tau(H^0(Y, \mathbb{Z}))$$

is projection.

Pimsner's Theorem A then allows us to easily calculate traces of $K_0(C^*(T^n, \mathbb{Z}))$ for any ergodic affine transformation θ of the n-torus, as pointed out to us by W. Paschke, thus extending Theorem 3.3.

COROLLARY C. *Let θ: $T^n \to T^n$ be an affine transformation of the n-torus which is ergodic with respect to Haar measure ν. Then*

$$\tau_\nu^*(K_0(C^*(T^n, \mathbb{Z}))) = \phi^{-1}(E),$$

where $E \subset S$ is the set of eigenvalues for θ and ϕ: $\mathbb{R} \to \mathbb{R}/\mathbb{Z}\ S^1$ is the natural projection.

Proof. Since T^n is connected $H^0(T^n, \mathbb{Z}) = \mathbb{Z}$ and we need only concern ourselves with $\ker(\theta^* - \mathrm{Id}) \subset H^1(T^n, \mathbb{Z})$. But an element of the kernel for $\theta^* - \mathrm{Id}$ corresponds to an element of the kernel of $(M^t - \mathrm{Id})$ in $\mathbb{Z}^n$, where $M \in GL(n, \mathbb{Z})$ is determined by θ. Hence an element of $\ker(\theta^* - \mathrm{Id}) \subset H^1(T^n, \mathbb{Z}) \simeq \mathbb{Z}^n$ corresponds to a topological eigenfunction for the transformation θ. Conversely any (measure theoretic) eigenfunction of (T^n, θ) is a.e. equal to an element of $\hat{T}^n$ and corresponds to an element of $\ker(M^t - \mathrm{Id}) \subset \mathbb{Z}^n$[11]. But then any eigenfunction for (T^n, θ) corresponds to an element of $\ker(\theta^* - \mathrm{Id}) \subset H^1(T^n, \mathbb{Z})$. In this case it is easy to see that

$\bar{A}_\nu(\ker(\theta^* - \mathrm{Id})) = E$, where $\bar{A}_\nu$ is as in the preceding remark. Hence $\tau^*(K_0(T'', \mathbb{Z})) = \phi^{-1}(E)$, as desired. ∎

Finally we indicate here an approach towards visualizing how elements with positive trace in $K_0(C^*(Y, \mathbb{Z})$ arise. Suppose that $\alpha > 0$ is in the range $\tau^*(K_0(C^*(Y, \mathbb{Z})))$ and determine $h \in \mathrm{Log}(Y, T)$ with $\int_Y h(y)\, dv = \alpha$. Form the induced flow $(X, \mathbb{R})$. As in the proof of Theorem B the function h given rise to a function g in $G(X)$ with

$$A_\mu([g]) = \int_Y h(y)\, dv = \alpha > 0.$$

By results of Schwarzman, if $A_{\mu'}([g]) > 0$ for every invariant measure μ' on X there will be a cross section K_h to the flow $(X, \mathbb{R})$ with $[g_{[K_h]}] = [g] \in H^1(X, \mathbb{Z})$. Such a cross section will exist if and only if $\int_Y h(y)\, dv' > 0$ for every T-invariant measure v' on Y. This will happen if h is a positive function on Y, for instance. In such a case, the results above show that there is a cross section K_h to $(X, \mathbb{R})$ and an action of $\mathbb{Z}$ on K_h such that

$$C^*(K_h, \mathbb{Z}) = p_h C^*(X, \mathbb{R})\, p_h$$

for some projection $p_h \in C^*(X, \mathbb{R})$ with $\tau_\mu(p_h) = \alpha$. Hence there is a strong Morita equivalence

$$C^*(Y, \mathbb{Z}) - p_1 C^*(X, \mathbb{R})\, p_h - C^*(K_h, \mathbb{Z})$$

which gives rise to a projection in some $M_n(C^*(Y, \mathbb{Z}))$ with trace α.

In this situation, in a natural way one can think of certain projections in $M_n(C^*(Y, \mathbb{Z}))$ as arising from topological dynamical systems (K, Z) which are "flow-equivalent" to $(Y, \mathbb{Z})$, i.e., whose induced flows are topologically conjugate to $(X, \mathbb{R})$, the induced flow for $(Y, \mathbb{Z})$. Such systems need not necessarily be conjugate to $(Y, \mathbb{Z})$; for example, irrational rotations of the circle by α and β are flow equivalent if and only if there exists $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2, \mathbb{Z})$ with $\alpha = (a\beta + b)/(c\beta + d) \bmod 1$. Thus in many cases finding projections in $C^*(Y, \mathbb{Z})$ comes down to finding systems which are flow equivalent to $(Y, \mathbb{Z})$. This is in itself a fascinating problem in topological dynamics (see [32]).

Note added in proof. After writing this Appendix we received a copy of R. Exel's thesis (U. C. Berkeley) and an article by N. Clarke, "A logarithm associated with a trace and a rotation numbers for automorphisms of C^*-algebras," which contains some related results obtained by different methods.

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